ON THE THEORY OF CONTROLLABILITY AND OBSERVABILITY OF LINEAR DYNAMIC SYSTEMS

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We consider one possible interpretation of the conditions which determine the optimum regenerating signal in a linear observable system.

1. In the theory of optimal processes two problems, among others, play an essential role.

1. The problem of control, that is, the problem of the choice of forces which carry a controlled object from one given state into another.

2. The problem of observation, that is, the problem of the operation which determines the unknown present (time varying) coordinates of the object in terms of allowed observable quantities.

These problems have been examined, in particular, within the bounds of the theory of optimum processes based on the maximum principle of Pontriagin [1], and from the standpoint of the theory of dynamic programing of Bellman [2]. The problems of the observation and control of linear systems under the condition of the minimum of a quadratic quality criterion were studied by Kalman [3], whereby a duality between the problems of control and observation were established. A number of papers have been devoted to applied problems of control and observation. (See, for example, [4]).

One of the possible approaches to the linear problems 1 and 2 is associated with the theory of the L-problem of moments [5]. Such an approach to the problem of control was suggested in [6]. The idea of this method is to interpret the problems of the computation of control forces or computation of regenerating signals as problems of the construction of linear functionals which take on given values on certain known elements. In this process the problem must be transformed in such a way that the bounds or the estimates to be minimized, can be expressed in terms of norms of the unknown functional. The goal of the present paper is to interpret, in the sense of the problem of observation and from the standpoint of the duality principle, those relationships which determine the optimum solution of problems 1 and 2 by an approach to these problems that uses the method of the L-problem of moments.

2. We examine the linear dynamic system

$$\frac{dx}{dt} = A(t)\,x + B(t)\,u \tag{2.1}$$

where x is the n-dimensional state vector of the coordinates of the controlled object; u is the r-dimensional vector of the control forces; $A(t)$, $B(t)$ are continuous or piecewise-continuous matrix functions of the corresponding dimensions; and t is time.

Note 2.1. In Equation (2.1) the coordinates $u_1, \ldots, u_r$ of the vector u are either force quantities actually applied to the object or u_i are quantities associated with the applied control forces and introduced into Equation (2.1) in a form convenient for the investigation of the problem.

We examine first the problem of control.

Problem 2.1. We have given an initial state $\{t^\circ, x^\circ\}$ of the object and a manifold M of its finite states $\{t', x'\}$, described by the parametric equations

$$x = f\,[z], \qquad t = t^\circ + \vartheta\,[z] \tag{2.2}$$

where z is a k-dimensional vector of the parameters limited, say, by certain condition which we write symbolically as

$$z \in Z \tag{2.3}$$

In addition there may be given supplementary limits and constraints.

The problem consists of determining the values of z° and of the function $u(t)(t^\circ \leqslant t \leqslant t^\circ + \vartheta)$ which satisfy the given limits and which are such that for $u = u(t)$ there exists a motion $x(t)$ of the system (2.1) satisfying equations

$$x(t^\circ) = x^\circ, \qquad x(t') = f\,[z^\circ], \qquad t' = t^\circ + \vartheta\,[z^\circ] \tag{2.4}$$

For completeness, we describe the approach to the solution of the problem 2.1. The solution $x(t)$ of Equation (2.1) we write in the form

$$x(t) = X\,[t^\circ, t]\,x^\circ + \int_{t^\circ}^{t} X\,[t^\circ, t]X^{-1}\,[t^\circ, \tau]\,B(\tau)\,u(\tau)\,d\tau \tag{2.5}$$

where X is the fundamental matrix of the solutions of Equation (2.1) for $u \equiv 0$.

We set $x(t) = f\,[z]$ and $t = t^\circ + \vartheta\,[z]$ into (2.5) and transform the obtained equation into the form

$$\int_{0}^{\vartheta[z]} G\,[z, t^\circ, \tau]\,v(z, t^\circ, \tau)\,d_\tau\zeta(z, t^\circ, \tau) = c\,(z) \tag{2.6}$$

or, in terms of coordinates,

$$\left\{\int_{0}^{\vartheta} G\,[z, t^\circ, \tau]\,v\,[z, t^\circ, \tau]\,d_\tau\zeta\,(z, t^\circ, \tau)\right\}_i = c_i(z) \tag{2.7}$$

so that the left side of (2.7) can be interpreted as the values of a certain linear functional φ (generated by Functions v and ζ) on the known elements $g^{(i)}(z, t^\circ, \tau)$ $(0 \leqslant \tau \leqslant \vartheta)$, determined by the matrix G, and so that the given conditions can be expressed in the form of the bound

$$\|\varphi\|^* \leqslant 1 \tag{2.8}$$

on the norm $\|\varphi\|^*$ of the functional φ. Here the quantities $v(z, t^\circ, \tau)$ and $d_\tau\zeta(z, t^\circ, \tau)$ coincide with the quantities $u(\tau)$ and $d\tau$, respectively, or are associated with them by a certain transformation which is introduced for the purpose of interpreting the given bound in the form of (2.8). The problem 2.1 has been reduced to the construction of a functional $\varphi[t^\circ, g(\tau)]$, which satisfies conditions (2.8)

$$\varphi[t^\circ, g^{(i)}(z, t^\circ, \tau)] = c_i(z) \qquad (i = 1, \ldots, n) \tag{2.9}$$

We shall denote the scalar product of the vectors q and l by the symbol $q\cdot l$. For the fixed z the problem (2.9) has the solution $\varphi[t^\circ, g(\tau)]$ if, and only if, [5]

$$\alpha(t^\circ, z) = \min_l \left(\left\| \sum_{i=1}^n l_i g^{(i)}(z, t^\circ, \tau)\right\|\right) > 0 \quad \text{for} \cdot c(z)\, l = 1 \tag{2.10}$$

where $\|g(\tau)\|$ is the norm in the functional space $\{g(\tau)\}$ $(0 \leqslant \tau \leqslant \vartheta)$ on which the functional φ is defined and which contains, in particular, the $g^{(i)}$ elements. In this case the minimum norm $\|\varphi^\circ\|^*$ of the functional φ, satisfying the conditions (2.9), is determined by Equation

$$\|\varphi^\circ\|^* = \frac{1}{\alpha(z, t^\circ)} \tag{2.11}$$

and the functional φ° itself satisfies the condition

$$\varphi^\circ[t^\circ, g^\circ] = \max_\varphi(\varphi[t^\circ, g^\circ]) = 1 \quad \text{for } \|\varphi\|^* = \alpha^{-1} \tag{2.12}$$

Here $g^\circ = \sum l_i^\circ g^{(i)}$ is the solution of the problem (2.10). Hence the problem 2.1 under the bound (2.8) is soluble if, and only if,

$$\alpha(t^\circ) = \max_z \alpha(t^\circ, z) \geqslant 1 \quad \text{for } z \in Z \tag{2.13}$$

or

$$\sup_z \alpha(t^\circ, z) > 1 \quad \text{for } z \in Z \tag{2.14}$$

if the upper side $\alpha(t^\circ, z)$ is not attained for $z \in Z$.

N o t e 2.2. In considering the problems (2.8) and (2.9) we have referred to [5]. In [5] it was assumed that the elements $g^{(1)}, \ldots, g^{(n)}$ are linearely independent. Here the linear independence of $g^{(i)}$ is not assumed. However this does not prevent from the use of the results of [5]. Indeed, we shall verify, for example, the sufficiency of conditions (2.10). Let the condition (2.10) be satisfied. We assume for definiteness that the elements $g^{(1)}, \ldots, g^{(m)}$ are linearly independent and that

$$g^{(i)} = \sum_{j=1}^m \beta_{ij} g^{(j)} \quad (i = m+1, \ldots, n) \tag{2.15}$$

The condition (2.10) clearly may be satisfied only under the condition

$$c_i = \sum_{j=1}^m \beta_{ij} c_j \qquad (i = m+1, \ldots, n) \tag{2.16}$$

whereby the following equality is valid

$$\min_l \left(\left\|\sum_{i=1}^n l_i g^{(i)}\right\|, \sum_{i=1}^n l_i c_i = 1\right) = \min_l \left(\left\|\sum_{j=1}^m l_j g^{(j)}\right\|, \sum_{j=1}^m l_j c_j = 1\right) \tag{2.17}$$

From (2.15) and (2.16) it follows that it is sufficient to satisfy the the conditions (2.9) only for $j = 1, \ldots, m$, since for $i > m$ these conditions are automatically satisfied. Under conditions (2.10), the conditions (2.9) for $i = 1, \ldots, m$ may be satisfied in accordance with the results of [5] since the elements $g^{(1)}, \ldots, g^{(m)}$ are linearly independent. The assertion on the sufficiency of conditions (2.10) for the solvability of the problem (2.9) without an assumption on the independence of $g^{(i)}$ has been proved, and further, as a consequence (2.17), Equation (2.11) has also been proved. In an analogous way the necessity of conditions (2.10) can be verified. We remark that the theorem on the solvability of the L-problem cited in [11] on page 100 is not accompanied by reservations on the linear independence of the $g^{(i)}$ elements

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2.3. The scheme of reducing the problem 2.1 to the L-problem of moments remains in force without essential changes in the case when not only the finite values $x(t')$ are constrained by the conditions laid down on the manifold M, but also when the values $x(t_j)$ at an arbitrary other instant of time are so constrained. In this case the manifold M is determined by the system of Equations

$$x = f^{[j]}[z], \qquad t_j = t^\circ + \vartheta^{[j]}[z] \qquad (j = 1, \ldots, \delta)$$

and likewise the equation $x(t_j) = f^{[j]}[z]$ must be fulfilled.

2.4. In the comparison of the problem 2.1 with the problem of observation we shall assume, for reasons of simplifying the calculations, that in (2.1) $u(t)$ is a scalar quantity, $B(t) = b(t)$ is a n-vector, and we shall limit ourselves in the L-problem to the simplest functional spaces, that is, in equations (2.7) we shall assume that

$$v(z, t^\circ, \tau) = u(\tau), \; d_\tau \zeta(z, t^\circ, \tau) = d\tau \text{ or } v \equiv 1, \; u(\tau)\, d\tau = d_\tau \zeta(\tau).$$

Then the elements $g^{(i)}$ and the quantity c_i in (2.7) and (2.9) have the form

$$g^{(i)}(z, t^\circ, \tau) = \sum_{j=1}^{n} \sum_{k=1}^{n} x_{ij}(t^\circ, t^\circ + \vartheta)\, x_{jk}^{-1}(t^\circ, t^\circ + \tau)\, b_k(t^\circ + \tau) \qquad (2.18)$$

$$c_i(z) = f_i[z] - \sum_{j=1}^{n} x_{ij}(t^\circ, t^\circ + \vartheta)\, x_j^\circ \qquad (2.19)$$

where x_{ij} and x_{ij}^{-1} are the corresponding elements in the matrices X and X^{-1}.

2.5. The maximum (2.13) is attained on Z when the functions $c_i(z)$ and $\vartheta[z]$ are continuous and Z is a bounded closed set. The validity of this assertion follows from the inequality

$$\limsup \alpha(t^\circ, z) \leqslant \alpha(t^\circ, z_0) \qquad \text{for } c(z) \to c(z_0), \; v[z] \to v[z_0]$$

which the function $\alpha(t^\circ, z)$ satisfies.

3. We consider the problem on observation. We are given the system

$$dx / dt = Q(t)\, x \qquad (3.1)$$

$$y = H(t)\, x \qquad (3.2)$$

where x is a n-dimensional vector of the state coordinates of the object; y is a m-dimensional vector of the observed coordinates; $Q(t)$, $H(t)$ are continuous or piecewise-continuous matrix-functions, and Equations (3.2) do not have a single-valued solution with respect to the vector x.

P r o b l e m 3.1. We have given the number $\vartheta > 0$ and the manifold N of quantities $\gamma[x, z]$ from which it is required to choose the quantity $\gamma(t+\vartheta)$, subject to determination by observation of the vector $y(t+\tau)$ on the time interval $0 \leqslant \tau \leqslant \vartheta$.

Let the manifold N be described parametrically by Equation

$$\gamma[x, z] = p(z) \cdot x \tag{3.3}$$

where z is a k-dimensional vector constrained, say, by some condition (2.3).

The problem consists of determining a linear functional $\varphi[t, y(\tau)]$, which is defined on m-dimensional vector-functions $y(\tau)$ $(0 \leqslant \tau \leqslant \vartheta)$ and which for every value of time $t \in T$ under consideration satisfies the condition

$$\varphi[t, y(t+\vartheta)] = p(z) \cdot x(t+\vartheta) \tag{3.4}$$

where $x(t)$ is the motion of the system (3.1), $z \in Z$ and the vector $y(t)$ is determined by Equation (3.2). Additional conditions may be given.

N o t e 3.1. The function $y(t+\tau)$ $(0 \leqslant \tau \leqslant \vartheta)$ which is observed may be associated with the motion $x(t+\tau)$ by a more complicated linear relation than (3.2.). For example one may give the relation

$$y(t+\tau) = Y(t, t+\tau)\, y(t) + \int_0^\tau G[t, \tau, t+\zeta]\, x(t+\zeta)\, d\zeta \tag{3.5}$$

which comes from a differential association $dy/dt = D(t)\, y + C(t)\, x$. The arguments which are adduced below may be easily extended to cases similar to (3.5).

If the given conditions are reduced to the restriction (2.8) of the norm of the functional φ (3.4), or to the requirement $\|\varphi\|^* = \min$ for each $t \in T$, or to the requirement $\sup_t (\|\varphi\|^*) = \min$ for some norm $\|\varphi\|^*$, then the problem 3.1 reduces to the L-problem. In fact, the solution $x(t)$ of Equation (3.1) satisfies the equality $x(t+\tau) = F(t, t+\tau)\, x(\tau)$, where $F(t^\circ, t)$ is the fundamental matrix of the solutions of the system (3.1). Therefore Equation (3.4) reduces to the equation

$$\varphi[t, H(t+\tau)\, F(t, t+\tau\ F^{-1}(t, t+\vartheta)\, x(t+\vartheta)] = p(z)\, x(t+\vartheta) \tag{3.6}$$

Taking into account the linear character of the functional φ and comparing coefficients for $x_i(t+\vartheta)$ on the left and right sides of (3.6), we obtain, as above, the system of Equations (2.9), where $t^\circ = t$ and

$$c_i(z) = p_i\ \ ) \qquad (i = 1, \ldots, n) \tag{3.7}$$

whereas the elements $g^{(i)}(z, t, \tau)$ are expressed in a known way by H, F and F^{-1}. Hence problem (3.4) for fixed z and t is solvable if, and only if, condition (2.10) is fulfilled (for $c(z) = p(z)$). Necessary and sufficient conditions for the solvability of problem 3.1 under the restriction (2.8) for given $t^\circ = t$ are given by the inequality (2.13) or by the inequality (2.14). The minimum norm $\|\varphi^\circ\|^*$ of the functional φ°, solving the problem (3.4) for given t and z, is given by Equation (2.11). Necessary and sufficient conditions for the solvability of problem 3.1 under the restriction (2.8) for all $t \in T$ is determined by the inequalities

$$\max_z \inf_t \alpha(t, z) \geqslant 1 \quad \text{or} \quad \sup_z \inf_t \alpha(t, z) > 1 \tag{3.8}$$

For simplicity of calculations we restrict ourselves to a further case where y is a scalar and the norm $\|\varphi\|^*$ relates to one of the standard functional spaces. Then $y = h(t)\, x$ where h is a n-dimensional vector. The $g^{(i)}$ elements in conditions (2.9), in this case are determined by Equations

$$g^{(i)}(z, t, \tau) = \sum_{j=1}^{n} \sum_{k=1}^{n} f_{ji}^{-1}(t, t + \vartheta)\, f_{kj}(t, t + \tau)\, h_k(t + \tau) \tag{3.9}$$

where f_{ij} and f_{ij}^{-1} are elements of the matrices F and F^{-1} respectively. If the matrices A, Q, B and H in Equations (2.1), (3.1) and (3.2) are associated by the relationships

$$Q = -A^*, \qquad H = B^* \tag{3.10}$$

where the symbol * denotes transposition, then for $t = t^\circ$ the elements (2.18) and (3.9) coincide, which follows from known properties of linear systems in [7]. Thus the problem 3.1 on the observation of the quantity $p(z) \cdot x(t + \vartheta)$ by means of the function $y(t + \tau)$ for the system (3.1), (3.2) under conditions (3.10) is equivalent to problem 2.1 on the control of the system (2.1) from the point $t^\circ = t,\ x(t^\circ) = x^\circ$ to the point $x(t^\circ + \vartheta) = f[z]$, if the vectors $p(z), x^\circ$ and $f[z]$ are associated by the relationship

$$p(z) = f[z] - X'(t^\circ, t^\circ + \vartheta)\, x^\circ.$$

This assertion is the expression of the duality principle of Kalman in [3]. The use of the L-problem in combination with intermediate transformations allows one to consider the duality relationship for a rather wide range of problems with various typical restrictions and relations.

4. We examine the interpretation of condition (2.10) of the solvability of problems 2.1 and 3.1 and also give an interpretation of conditions (2.11) to (2.14). We first introduce some definitions and notations. Let the function $\eta(\tau)$ be defined on the interval $0 \leqslant \tau \leqslant \vartheta$ and let this function be considered as an element of a certain functional space with norm $\rho(\eta)$. We shall call the quantity $\rho(\eta)$ the intensity of the signal $\eta(\tau)$. We shall assume that the class of functions $\{\zeta(\tau)\}$ $(0 \leqslant \tau \leqslant \vartheta)$ defines the space in [8] of linear functionals $\varphi(\eta)$, defined on the functions $\eta(\tau)$ with norm $\rho(\eta)$. The norm of the function $\zeta(\tau)$, which is equal to the norm of the corresponding functional φ, we shall call the intensity of the signal $\zeta(\tau)$ conjugate to $\rho(\eta)$, and shall denote it by the symbol $\rho^*(\zeta)$. On the other hand if the function $\eta(\tau)$ is chosen from a class of functions defining linear functionals $\varphi(\xi)$ with norm $\rho(\eta)$ on some space $\{\xi(\tau)\}$ then the norm of the function $\xi(\tau)$ we shall call the intensity of the signal $\xi(\tau)$, the original for $\rho(\eta)$, and shall denote it by $\rho_*(\xi)$.

We examine the observable system (3.1) and (3.2). Let there exist the possibility of measuring the quantity $y(t)$ with an error $w(t)$ arising from some noise. The quantity $r(t) = y(t) + w(t)$ we shall call the complete signal, the quantity $w(t)$ we shall call the noise, and the function $y(t)$ (3.2) we shall call the useful signal. We assume that an exact value of the function $w(t)$ is unknown, but that a class of allowable functions$\{w(t)\}$ has been determined and that we have been given an estimate of some sort of intensity $\rho(w)$ of the signal $w(t+\tau)(0 \leqslant \tau \leqslant \vartheta)$. We assume for definiteness that $w(t)$ is a continuous function and that

$$\rho(w) = \max_\tau |w(t+\tau)| \quad \text{for} \quad t \in T \quad \text{and} \quad 0 \leqslant \tau \leqslant \vartheta \tag{4.1}$$

whereby the bound is given

$$\rho(w) \leqslant \delta \quad (\text{for } t \in T) \tag{4.2}$$

N o t e 4.1. The considerations which follow below also carry through in the general case of $\rho(w)$ and may be made specific for other typical intensities $\rho(w)$, for example for the case

$$\rho(w) = \left[\int_0^\vartheta |w(t+\tau)|^p d\tau\right]^{1/p}$$

In this case the conjugate intensities $\rho^*(w)$ should change accordingly with the change in the character of $\rho(w)$.

If there are no additional restrictions on the operation φ, which solves problem 3.1, then it is natural to put a question about the determination of an operation computing $p(z)\cdot x(t+\vartheta)$, such, that in the presence of noise $w(t)$ (4.2) gives the smallest absolute error. The resolving operation

$$\varphi[t, r(t+\tau)] = p(z)\cdot x(t+\vartheta) + \omega_w \qquad (\omega_w - \text{error}) \tag{4.3}$$

is sought in the form of a linear functional

$$\varphi[t, r(t+\tau)] = \int_0^\vartheta r(t+\tau)\, d_\tau \zeta(z, t, \tau) \tag{4.4}$$

generated by the function $\zeta(z, t, \tau)$ $(0 \leqslant \tau \leqslant \vartheta)$ with the norm

$$\|\varphi^*\| = \rho^*(\zeta) = \int_0^\vartheta |d_\tau \zeta(z, t, \tau)| \tag{4.5}$$

where for every fixed space z and every value of t the expression $\zeta(z, t, \tau)$ is a function of τ with bounded measure. In this case

$$\int_0^\vartheta y(t+\tau)\, d_\tau \zeta(z, t, \tau) = p(z)\cdot x(t+\vartheta) \tag{4.6}$$

$$\int_0^\vartheta w(t+\tau)\, d_\tau \zeta(z, t, \tau) = \omega_w \tag{4.7}$$

$$\rho^*(\zeta) = \sup_w \left|\int_0^\vartheta w(t+\tau)\, d_\tau \zeta(z, t, \tau)\right| \quad \text{for} \quad \rho(w) = 1 \tag{4.8}$$

We shall call the quantity $\Delta_{z,t} = \sup_w |\omega_w|$ the absolute error of the operation φ under the condition $\rho(w) \leqslant \delta$. In accordance with (4.5) to (4.8) we have $\Delta_{z,t} = \delta\rho^*(\zeta)$. Thus the problem of the choice of the operation φ which gives the smallest error $\Delta_{z,t}$, for a given z and t means that it is necessary to find a function $\zeta(z, t, \tau)$, which satisfies condition (4.6) and which has the smallest conjugate intensity $\rho^*(\zeta)$ (4.5). Hence let the quantity z be fixed at the outset.

On the ground of the results cited in Sections 2 and 3 we obtain the following result. We denote by the symbol $y(l, t, \tau)$ the quantity

$$y(l, t, \tau) = \sum_{i=1}^{n} l_i g^{(i)}(z, t, \tau) \tag{4.9}$$

where the quantities $g^{(i)}$ are determined by the equality (3.9). The solution signal $\zeta(z, t, \tau)$ (4.6) exists for every $t \in T$ if, and only if

$$\min_l [\rho(y(l, t, \tau))] = \alpha(t, z) > 0 \qquad \text{for } l \cdot p(z) = 1,\ t \in T \tag{4.10}$$

The smallest attainable error $\overset{\circ}{\Delta}_{z,t}$ is given by the quantity

$$\overset{\circ}{\Delta}_{z,t} = \frac{\delta}{\alpha(t, z)} \tag{4.11}$$

and the optimum solution signal $\zeta^\circ(z, t, \tau)$ satisfies the condition

$$\int_0^\vartheta y(l^\circ, t, \tau)\, d_\tau \zeta^\circ(z, t, \tau) = \max_\zeta = 1 \qquad \text{for } \rho^*(\zeta) = \frac{1}{\alpha(t, z)} \tag{4.12}$$

where l° is the solution of problem (4.10).

The results that have been obtained can be visualized in the following manner. We shall say that the useful signal $y(t + \tau)$ $(0 \leqslant \tau \leqslant \vartheta)$ carries the quantity $p(z) \cdot x(t + \vartheta) = \gamma$, if it is generated by a motion $x(t)$ of the system (3.1) in accordance with (3.2) which satisfies the condition

$$p(z) \cdot x(t + \vartheta) = \gamma.$$

Then the quantity $y(l, t, \tau)$ (4.9) in problem (4.10) is none other than the useful signal $y(t + \tau)$, carrying the unknown quantity $p(z) \cdot x(t + \vartheta) = 1$. Hence the problem (4.6) on the computation of $p(z) \cdot x(t + \vartheta)$ by means of $y(t + \tau)$ (3.2) is solvable if, and only if, the smallest intensity

$$\rho(y(t + \tau))$$

of the signal $y(t + \tau)$, carrying the quantity $p(z) \cdot x(t + \vartheta) = 1$, remains positive for all $t \in T$.

Now let the vector z be chosen from a bounded closed set Z, $p(z)$ depend continuously on z and T is the interval $t_1 \leqslant t \leqslant t_2$. Then the following conclusion is valid. If there exists such a $z \in Z$, for which the smallest intensity $\rho(y(t + \tau))$ of the signal $y(t + \tau)$, carrying the quantity

$$p(z) \cdot x(t + \vartheta) = 1,$$

remains positive for all $t \in T$, then there exists an optimum solution operation φ generated by the function $\zeta^\circ(t, \tau)$. This optimum operation gives the smallest absolute error

$$\Delta = \min_z \max_t \Delta_{z,t} = \frac{\delta}{\alpha}, \qquad \alpha = \max_z \min_t \alpha(t, z) \tag{4.13}$$

and satisfies the conditions

$$\int_0^{\vartheta} y\,(l^{\circ},\, t,\, \tau)\; d_{\tau}\zeta^{\circ}\,(t,\, \tau) = 1 = \max_{\zeta} \qquad \text{for } \|\zeta\|^{*} = \rho^{*}\,(\zeta) = \frac{1}{\alpha}$$

where l° is the solution of the problem (4.10) for $t \in T$ for z solving the problem (4 13). The result that has been obtained with account taken of the principle of duality between problems (2.1) and (3.1) can be formulated in general form in the following manner.

The signal $y\,(t + \tau)$, carrying the quantity $p\,(z)\cdot x\,(t + \vartheta) = 1$ and having the smallest possible intensity $\rho\,(y)$, we shall call the minimum and shall denote it by the simbol $[y^{\circ}\,(t + \tau)\,|\,\gamma = 1,\, \rho]$.

T h e o r e m 4.1. The problem 3.1 on the observation of the quantity $p\,(z)\cdot x\,(t + \vartheta)$ by means of the signal $y\,(t)$ (3.2) has a solution if, and only if, it is possible to find $z \in Z$, for which the intensity $\rho\,(y^{\circ})$ of the minimum signal $[y^{\circ}\,(t + \tau)\;\gamma = 1,\, \rho]$ is different from zero for all $t \in T$. If in this case there is given the bound $\rho\,(w) \leqslant \delta$ on the intensity of the noise $w(t)$, then the lower limit Δ° of the error Δ in the determination of the quantity $p\,(z)\cdot x\,(t + \vartheta)$ by means of the complete signal $r\,(t) = y\,(t) + w\,(t)$ is determined by Equation

$$\Delta^{\circ} = \delta\,/\,\alpha, \qquad \alpha = \sup_{z}\inf_{t}\alpha\,(t,\, z)$$

Here $\alpha\,(t,\, z)$ is the intensity $\rho\,(y^{\circ})$ of the mimimum useful signal

$$[y^{\circ}\,(t + \tau)\,|\,\gamma = 1,\, \rho].$$

In the case when $\sup_{z}\inf_{t}\alpha\,(t,\, z)$ is attained on Z, there exists an optimum solution operation φ° for which $\Delta = \Delta^{\circ}$. Besides, the optimum solution signal $\zeta^{\circ}(t,\, \tau)$ has the conjugate intensity $\rho^{*}(\zeta^{\circ}) = 1\,/\,\alpha$ and is distinguished from all other signals ζ with intensity $\rho^{*}(\zeta) = 1\,/\,\alpha$ by the property that on the useful minimum signal $[y^{\circ}\,(t + \tau)\,|\,\gamma = 1,\, \rho]$ the operation φ°, generated by the function ζ°, gives the largest possible value.

T h e o r e m 4.2. Let there be given problem 2.1 on the control of system (2.1) from the point x°, t° on the manifold M (2.2) under the condition that the intensity of the control signal $u\,(t^{\circ} + \tau)$ $(0 \leqslant \tau \leqslant \vartheta)$ (or the signal ζ, where $d_{\tau}\zeta = ud\,\tau$) is restricted by the inequality $\rho\,(u) \leqslant 1$ (or $\rho\,(\zeta) \leqslant 1$).

Problem 2.1 has a solution if, and only if, it is possible to chose $z \in Z$, satisfying the following condition: the original intensity $\rho_{*}\,(y^{\circ})$ of the minimum signal $[y^{\circ}\,(t^{\circ} + \tau)\,|\,\gamma = 1,\, \rho_{*}]$ $(0 \leqslant \tau \leqslant \vartheta\;[z])$ of the adjoint observable system $dx\,/\,dt = -A^{*}\,(t)\,x,\; y = B^{*}x$ satisfies the condition $\rho_{*}\,(y^{\circ}) \geqslant 1$, and $p\,(z) = f\,[z] - X\,(t^{\circ},\, t^{\circ} + \vartheta)\,x^{\circ}$.

For such a choice z the optimum control signal $u^{\circ}(t^{\circ} + \tau)$ (or $\zeta^{\circ}(t^{\circ},\, \tau)$) has the intensity

$$\rho\,(u^{\circ}) = 1\,/\,\alpha\,(z), \qquad \alpha\,(z) = \rho_{*}\,[(y^{\circ}\,(t^{\circ} + \tau)\,|\,\gamma = 1,\; \rho_{*}])$$

and it is distinguished from all other signals with intensity $\rho = 1/\alpha(z)$ by the property that on the minimum signal $[y^\circ(t^\circ + \tau) | \gamma = 1, \rho_*]$ the operation φ°, generated by function u° (or ζ°), has the largest possible value.

Thus the optimum signal $\zeta^\circ(t, \tau)$ has the property that it produces the largest possible value in the most unfavorable case of a useful signal $y(t + \tau)$ (from the point of view of the intensity $\rho(y)$). As has been shown above this property of the signal ζ° follows from the solution of problem 3.1, based on the theory of the L-moment problem. It is useful, however, to interpret the conclusions adduced above from another point of view.

5. We shall compare problem 3.1 with a game [9]. Let us examine problem 3.1 afresh, supposing z and t are fixed, and for definiteness, we shall assume that the intensities of the signals $y(t + \tau)$ and $\zeta(t, \tau)$ are defined by equalities

$$\rho(y) = \max_\tau |y(t + \tau)|, \qquad \rho^*(\zeta) = \int_0^\vartheta |d_\tau \zeta(t, \tau)| \tag{5.1}$$

G a m e 5.1. The strategy of the first player are the functions of $\zeta(\tau)$ with intensity $\rho^*(\zeta) \leqslant 1$, and the strategy of the second player are the functions of $y(t)$, satisfying the conditions

$$y(\tau) = h(\tau) \cdot x(\tau), \qquad x(\tau) = F(t + \vartheta, t + \tau)\, x(\vartheta)$$
$$p(z) \cdot x(\vartheta) = 1$$

The utility function μ is determined by Equation

$$\mu[\zeta, y] = \int_0^\vartheta y(\tau)\, d_\tau \zeta(\tau) \tag{5.2}$$

The goal of the first player is to obtain the largest possible value of μ, and the goal of the second player is to obtain the smallest possible value of μ.

It turns out that the game 5.1 has a saddle point [9] (ζ°, y°), that is $\mu[\zeta^\circ, y^\circ] = \max_\zeta \min_y \mu = \min_y \max_\zeta \mu$, and besides

$$\mu[\zeta^\circ, y] \geqslant \mu[\zeta^\circ, y^\circ], \qquad \mu[\zeta, y^\circ] \leqslant \mu[\zeta^\circ, y^\circ] \tag{5.3}$$

Consequently, ζ° and y° are really the best possible strategies for the first and second players. The optimum strategy y° is naturally determined by the condition $\rho(y^\circ) = \min_y \rho(y) = \alpha(t, z)$, and the optimum strategy ζ° satisfies the condition $\mu[\zeta^\circ, y^\circ] = \max_\zeta \mu[\zeta, y^\circ]$ for $\rho^*(\zeta) = 1$. It follows, therefore, that the optimum strategy $y^\circ(\tau)$ of the game 5.1 coinsides with the minimum useful signal $[y^\circ(t + \tau) \mid \gamma = 1, \rho]$, and the optimum strategy $\zeta^\circ(\tau) = \alpha(t, z)\, \zeta^\circ(t, \tau)$, where $\zeta^\circ(t, \tau)$ is the optimum solution signal of problem 3.1. To clarify why just the function $\zeta^\circ(\tau)$ is the solution signal of the problem 3.1, that is why it gives the quantity

$p(z)\,x(\vartheta) = 1$, for all $y(\tau)$ and not larger as it is possible in accordance with (5.3), we shall note one property which the optimum strategy $\zeta^\circ(\tau)$ has in the present case. In the theory of games the theorem [9] is known: if q° and s° are the optimum mixed strategies of a certain game $\mu[q, s]$ and s is an arbitrary active strategy of the second player (i.e. s is a strategy contained in the mixed strategy s°), then the following equality is valid

$$\mu\,[q^\circ, s] = \mu\,[q^\circ, s^\circ] \tag{5.4}$$

It turns out that the construction of the game 5.1 in the known sense is sufficiently analogous to the construction of a game with mixed strategy that an arbitrary strategy $y(\tau)$ turns out to have the above indicated property of active strategies s. Hence the assertion is true.

T h e o r e m 5.1. The problem on the optimum signal $\zeta^\circ(t, \tau)$ which for the system (3.1) regenerates the quantity $p(z)\cdot x(t+\vartheta)$, by means of the signal $y(t+\tau)$ (3.2) under the condition of minimum intensity $\rho^*(\zeta)$ is equivalent to the problem on the choice of an optimum strategy $\zeta^\circ(\tau)$ in the game 5.1. The game 5.1 has a saddle point $(\zeta^\circ(\tau), y^\circ(\tau))$. The strategy $\zeta^\circ(\tau)$ coincides with the signal $\zeta^\circ(t, \tau)$ within a factor $\alpha(t, z)$, and the strategy $y^\circ(\tau)$ coincides with the minimum signal $[y^\circ(t+\tau)\,|\,\gamma = 1, \rho]$.

The strategy $\zeta^\circ(\tau)$ of the game 5.1 turns out to be the solution signal for the problem 3.1 for the following reason: an arbitrary strategy $y(\tau)$ of the game 5.1 relative to (ζ°, y°) has the property (5.4), which is characteristic for active strategies s of extended finite games [9].

N o t e 5.1. If the intensity $\rho^*(\zeta)$ is determined by equality (5.1), the analogue of the problem 3.1 to the theory of games may still be continued by a visual method. In fact, let us assume that the first player has only the pure strategies $\zeta^\pm(\tau, \tau_*)$ of the form $d_\tau\zeta^\pm(\tau, \tau_*) = \pm\,\delta(\tau - \tau_*)\,\alpha\tau$, where $\delta(\tau)$ denotes the delta functions and τ_* are every possible numbers from the interval $[0, \vartheta]$.

Then the optimum strategy $\zeta^\circ(\tau)$ may be interpreted as a mixed strategy, where the pure strategies $\zeta(\tau, \tau_*)$ enter with probability $|d_\tau\,\zeta^\circ(\tau_*)|$.

5.2. We consider the problem 2.1 on the optimum control [1] (for definiteness under the condition $\rho_\cdot(u) = \sup_\tau |u(\tau)| = \min$). The solution $u^\circ(\tau)$ of this problem is determined by the maximum principle [1] from which it follows that

$$(\psi^\circ, u^\circ) = \max H(\psi^\circ, u) = \max(\psi^\circ(\tau)\cdot bu(\tau)) \qquad \text{for } |u(\tau)| \leqslant \rho(u^\circ)$$

where $\psi^\circ(\tau)$ is the solution of Equation $d\psi/dt = -A^*(t)\,\psi$. By the duality principle the problem on observation 3.1 corresponds to the problem on control (2.1). In accordance with Theorem 5.1 the problem 3.1 is in turn equivalent to the game 5.1, where exactly

$$\mu\,[u(\tau),\ y(\tau)] = \int_0^\vartheta H d\tau = \int_0^\vartheta \psi(\tau)\cdot bu(\tau)\,d\tau, \qquad y(\tau) = \psi(\tau)\cdot b$$

Hence Teorem 5.1 shows, in particular for the class of problems considered, a possible interpretation of the maximum principle as conditions $\max_u \min_\psi \mu$ for a game selected in a corresponding way. In this case it is to be stressed that the element ψ°, which appears in the maximum principle, should satisfy the maximum condition for the game

$$\mu = \int_0^\vartheta H(\psi, u)\, d\tau$$

5.3. The connection between the problem on observation and the game 5.1 which has been examined in this section is explained by the fact that L-problem and the problem of the separation of sets upon which one usually investigates games with a saddle point are of analogous nature in terms of the formulation of the problem and in terms of the method of investigation.

6. We consider an example. Let it be required to stabilize the system

$$dx / dt = Ax + bu \tag{6.1}$$

by means of the control $u = p \cdot x$, whereby the quantities $r(t) = h \cdot x(t) + w(t)$, $|w(t)| \leqslant \delta$. are to be measured. In accordance with the procedure described in [10], one may successively solve the problem on the stabilization of system (6.1) and then the problem on the observation of the quantity $p \cdot x(t)$ by means of $y(t)$. Let the Equation $|a_{ij} + b_i p_j(z) - \lambda\delta_{ij}| = 0$ have, in all cases, the roots λ_k with $\operatorname{Re}\lambda_k < 0$ $(k = 1, \ldots, n)$, whenever the vector $p(z)$ changes within the limits

$$p(z) = p^\circ + Dz \qquad (-\varepsilon_j \leqslant z_j \leqslant \varepsilon_j, D = \{d_{ij}\},\ i = 1, \ldots, n;\ j = 1, \ldots, k) \tag{6.2}$$

Then in the process of the solution of the problem on the stabilization (6.1) there arises the problem [10] on the observation 3.1 as follows: for a chosen $\vartheta > 0$ find z from (6.2) and an operation $\varphi[r(\tau)]$ $(0 \leqslant \tau \leqslant \vartheta)$, which regenerates the quantity $p(z) \cdot x(t)$ by means of $r(t)$ and with the smallest error, where

$$\varphi[r(t+\tau)] = \varphi[y(t+\tau)] + \varphi[w(t+\tau)] = p(z)\, x(t+\vartheta) + \omega_w$$

$$dx / dt = Ax, \qquad y = h \cdot x$$

It was shown above that this problem reduces to the following problem: find and a function $\zeta^\circ(\tau)$ for which

$$\int_0^\vartheta g^{(i)}(\tau)\, d_\tau \zeta^\circ(\tau) = p_i(z), \quad \rho^*(\zeta) = \int_0^\vartheta |d_\tau \zeta^\circ(\tau)| = \min\nolimits_{\zeta, z}, \ g^*(\tau) = h^*(\tau)\, e^{A(\tau-\vartheta)} \tag{6.3}$$

(g^*, h^* are row vectors)

The problem is solvable, if, and only if, there exists a z from (6.2) for which $\alpha(z) = \min_y \max_\tau |y(\tau)| > 0$ for $x(\vartheta)\, p(z) = 1$. Let this condition be fulfilled. Then the impulse $\rho^*(\zeta^\circ)$ (6.3) of the optimum regenerating signal $\zeta^\circ(\tau)$ is determined by Equation

$$\rho^*(\zeta^\circ) = \frac{1}{\alpha}, \qquad \alpha = \max\nolimits_z \alpha(z) = \max\nolimits_z [\min\nolimits_l \max\nolimits_\tau |l \cdot g(\tau)|] \tag{6.4}$$

$$(0 \leqslant \tau \leqslant \vartheta,\ l \cdot p(z) = 1,\ -\varepsilon_i \leqslant z_i \leqslant \varepsilon_i,\quad i = 1, \ldots, k)$$

If by the conditions of the problem the value of the error $\Delta = \sup_w \omega_w$ is not allowed to exceed the number ν, then the problem is solvable if, and only if $\alpha \geqslant \delta / \nu$. Changing the scale, if necessary, we rewrite the latter condition in the form of an inequality

$$\alpha \geqslant 1 \tag{6.5}$$

N o t e 6.1. We assume that $p(z) \neq 0$ in (6.2), then $\sup_z \alpha(z) < \infty$ and the maximum (6.4) is actually attained.

Assuming that the matrix A is nonsingular and that the condition of general position is fulfilled [1,3], that is that the row vectors h^*, h^*A, ..., h^*A^{n-1} are linearly independent. Then the functions $g^{(i)}(\tau)$ are linearly independent and for an arbitrary $l \neq 0$ the expression $l \cdot g(\tau)$ has a maximum only at isolated points $\tau_j (i = 1, \ldots, s(l))$. Therefore $\zeta^\circ(\tau)$ satisfies condition

$$d\zeta^\circ(\tau) = \sum_{j=1}^{n} \varkappa_j \delta(\tau - \tau_j)$$

and the unknown operation $\varphi[r]$ has the form

$$\varphi[r(\tau)] = \sum_{j=1}^{n} \varkappa_j r(\tau_j)$$

Thus to solve problem 3.1 in the given case it is necessary to solve problem (6.4). The quantity $\max_\tau |l \cdot g(\tau)| = \rho(y)$, considered as a function of z and l under the conditions $p(z) \cdot l = 1$ and (6.2), has the saddle point (z°, l°).

This is verified by arguments which are analogous to those that are used to prove the maximum theorem for extended finite games [9]. Therefore the solution (z°, l°) of the problem (6.4) satisfies condition

$$z_i^\circ = -\varepsilon_i \operatorname{sign}\left(\sum_{j=1}^{n} d_{ji} l_j\right)$$

and the inequality (6.5) is equivalent to the inequality

$$\min_l \left[(\max_\tau |l \cdot g(\tau)|) + \sum_{i=1}^{k} \varepsilon_i \left|\sum_{j=1}^{n} d_{ji} l_j\right|\right] \geqslant 1 \qquad \text{for } l \cdot p^\circ = 1 \tag{6.6}$$

In particular, if $p_i(z) = p_i^\circ + z_i$, then condition (6.6) has the form

$$\min_l \left[(\max_\tau |l \cdot g(\tau)|) + \sum_{i=1}^{n} \varepsilon_i |l_i|\right] \geqslant 1 \qquad \text{for } l \cdot p^\circ = 1 \tag{6.7}$$

In this case the problem considered 3.1, coincides with the problem on the control of the system $dx/dt = -A^*x + hu$ from the point $x^\circ = 0$, $t^\circ = 0$ into $\{\varepsilon_i\}$, a vicinity of the point p°, by means of the force $u(t)$ whose impulse

$$\int_0^\vartheta |u(\tau)| \, d\tau$$

does not exceed unity. The condition of observability and controllability (6.7) can also be deduced in an other way, for example, by considering the problem (6.3) as an L-problem on the elements of the space

$$\{\eta_i, \ i = 1, \ldots, n; \ \eta(\tau), 0 \leqslant \tau \leqslant \vartheta\}$$

with norm

$$\|\eta\| = \sum |\eta_i| + \max_\tau \eta(\tau)|$$

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